Side Note: Perturbation scheme on $SO(3)$ Rotation

Reference: State estimation for robotics by Prof. Barfoot [1]
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Preliminary

For a more thorough treatment on Lie theory for state estimation in robotics, one may refer to:

- Barfoot's book: State Estimation For Robotics
- Chirikjian's book: Stochastic models, information theory, and Lie groups (Volume 1 & 2)
- Sola et al., (2018) A micro Lie theory for state estimation in robotics

1 Perturbation Applied to a Rotation on $SO(3)$

For a rotation in 3D, perturbations can be applied either on the Lie Algebra (i.e., the tangential space) or directly on the Special Orthogonal Group $SO(3)$ (i.e., on the manifold).

A more general discussion on the perturbation options is presented in Section 7.3.1 in [1]. Referring to [1], there are different perturbation options, which are left, middle and right perturbations. In this document, we will discuss the right and middle perturbations.

1.1 Perturbation on Lie Group (Right Perturbation)

For perturbation applied directly on the $SO(3)$ manifold, we let subscript a denote the original rotation and subscript b denote the perturbed rotation. We let $\exp(\boldsymbol{\phi}_m^\wedge)$ be a right perturbation applied to rotation matrix $\mathbf{R}_a$, and we have

$$\exp(\boldsymbol{\theta}_b^\wedge) = \mathbf{R}_b = \mathbf{R}_a \exp(\boldsymbol{\phi}_m^\wedge) = \exp(\boldsymbol{\theta}_a^\wedge)\,\exp(\boldsymbol{\phi}_m^\wedge), \qquad \boldsymbol{\theta}_b, \boldsymbol{\theta}_a,\, \boldsymbol{\phi}_m \in \mathbb{R}^3, \tag{1}$$

where $\exp(\cdot)$ is the matrix exponential map; $(\cdot)^\wedge$ is the skewness operator that returns the corresponding skew-symmetric matrix of a vector; and $\boldsymbol{\theta}_a$ is the corresponding angle-axis representation of the rotation matrix $\mathbf{R}_a$.

If we want to recover the angle-axis vector, $\boldsymbol{\theta}_b$ (i.e., the tangential vector in the tangential space), of $\mathbf{R}_b$, we use the matrix logarithm map as

$$\begin{aligned} \boldsymbol{\theta}_b &= \ln\left(\exp(\boldsymbol{\theta}_a^\wedge)\,\exp(\boldsymbol{\phi}_m^\wedge)\right)^\vee \\ &\approx \boldsymbol{\theta}_a + \mathrm{J}_r(\boldsymbol{\theta}_a)^{-1}\,\boldsymbol{\phi}_m, \end{aligned} \tag{2}$$

where $\mathrm{J}_r(\cdot)$ is the right jacobian of $SO(3)$ given as (see [1, Section 7.1] and [2]):

$$
\begin{aligned}
\mathrm{J}_r(\boldsymbol{\psi}) &= \frac{\sin\psi}{\psi}\,\mathbf{I}_{3\times3} + (1 - \frac{\sin\psi}{\psi})\,\mathbf{a}\mathbf{a}^T - \frac{1-\cos\psi}{\psi}\,\mathbf{a}^\wedge && \text{(3a)}\\
&\equiv \mathbf{I}_{3\times3} - \frac{1-\cos\psi}{\psi^2}\,\boldsymbol{\psi}^\wedge + \frac{\psi-\sin\psi}{\psi^3}(\boldsymbol{\psi}^\wedge)^2 && \text{(3b)}\\
&\approx \mathbf{I}_{3\times3} - \frac{1}{2}\,\boldsymbol{\psi}^\wedge, && \text{(3c)}
\end{aligned}
$$

where $\psi = \|\boldsymbol{\psi}\|$ is the angle of rotation with a unit of [rad]; $\mathbf{a} = \frac{\boldsymbol{\psi}}{\psi}$ is a 3×1 vector representing the axis of rotation; $\mathbf{I}_{3\times3}$ is the 3×3 identity matrix.

Note that in the actual implementation, one should be aware of the possible numerical instability due to the fact that ψ and its power in the denominator can be a quite small value (the perturbation is often associated with a small angle). Therefore, the approximation (3c) should be used when ψ (if (3a) is adopted) or ψ^3 (if (3b) is adopted) is smaller than the numerical limit of a computer to avoid dividing by a value close to zero.

The inverse of the right Jacobian is then given as (see [1, Section 7.1] and [2]):

$$
\begin{aligned}
\mathrm{J}_r(\boldsymbol{\psi})^{-1} &= \frac{\psi}{2}\,\cot\frac{\psi}{2}\,\mathbf{I}_{3\times3} + \left(1 - \frac{\psi}{2}\,\cot\frac{\psi}{2}\right)\mathbf{a}\mathbf{a}^T + \frac{\psi}{2}\,\mathbf{a}^\wedge && \text{(4a)}\\
&\equiv \mathbf{I}_{3\times3} + \frac{1}{2}\,\boldsymbol{\psi}^\wedge + \left(\frac{1}{\psi^2} + \frac{1+\cos\psi}{2\,\psi\,\sin\psi}\right)(\boldsymbol{\psi}^\wedge)^2 && \text{(4b)}\\
&\approx \mathbf{I}_{3\times3} + \frac{1}{2}\,\boldsymbol{\psi}^\wedge, && \text{(4c)}
\end{aligned}
$$

1.2 Perturbation on Lie Algebra (Middle Perturbation)

For a perturbation applied on Lie Algebra, $\mathfrak{so}(3)$, i.e., the tangential space of $SO(3)$, we let $\boldsymbol{\phi}_t$ be an angle-axis representation of a perturbation on $\mathfrak{so}(3)$, and we use subscript c to denote the original rotation and subscript d to mean the perturbed rotation. Thus,

$$
\begin{aligned}
\exp\left(\boldsymbol{\theta}_d^\wedge\right) = \mathbf{R}_d = \exp\left((\boldsymbol{\theta}_c + \boldsymbol{\phi}_t)^\wedge\right) &\equiv \exp\left(\boldsymbol{\theta}_c^\wedge + \boldsymbol{\phi}_t^\wedge\right)\\
&\approx \exp\left(\boldsymbol{\theta}_c^\wedge\right)\,\exp\left((\mathrm{J}_r(\boldsymbol{\theta}_c)\,\boldsymbol{\phi}_t)^\wedge\right) \qquad (5)\\
&= \mathbf{R}_c\,\exp\left(((\mathrm{J}_r(\boldsymbol{\theta}_c)\,\boldsymbol{\phi}_t)^\wedge)\right),
\end{aligned}
$$

where $\mathrm{J}_r(\boldsymbol{\theta}_c)$ is the right jacobian evaluated at $\boldsymbol{\theta}_c$ by using (3). Notice that the perturbation is introduced in the vector space, where $\boldsymbol{\phi}_t \in \mathbb{R}^3$.

To recover $\boldsymbol{\theta}_d$ from $\mathbf{R}_d$, we have

$$
\begin{aligned}
\boldsymbol{\theta}_d = \ln\left(\mathbf{R}_d\right)^\vee &= \ln\left(\exp\left(\boldsymbol{\theta}_c + \boldsymbol{\phi}_t\right)^\wedge)\right)^\vee \\
&= \boldsymbol{\theta}_c + \boldsymbol{\phi}_t
\end{aligned} \qquad (6)
$$

Therefore, depending on the perturbation scheme used, the form of the recovered angle-axis representation for the perturbed rotation would be slightly different.

In an optimization problem, if $\mathbf{R}_a$ or $\mathbf{R}_c$ represents an operating point of the rotation, and $\boldsymbol{\phi}_m$ or $\boldsymbol{\phi}_t$ represents the corresponding update, respectively. Then, if a perturbation is applied through right perturbation, the update to $\mathbf{R}_a$ should follow (1), and one can use (2) to recover the angle-axis representation of the rotation (if a vector form is required). In contrast, if a middle perturbation is to be applied, then one would obtain a perturbation vector $\boldsymbol{\phi}_t$ from an optimization algorithm and apply the perturbation to $\boldsymbol{\theta}_c$ through (6), and the perturbed rotation matrix, $\mathbf{R}_d$, is obtained by applying the exponential map over $\boldsymbol{\theta}_d^\wedge$.

References

[1] Timothy D Barfoot. *State estimation for robotics*. Cambridge University Press, 2017.

[2] Christian Forster, Luca Carlone, Frank Dellaert, and Davide Scaramuzza. On-manifold preintegration for real-time visual–inertial odometry. *IEEE Transactions on Robotics*, 33(1):1–21, 2016.